CANONICAL MODELS FOR THE FORWARD AND BACKWARD ITERATION OF HOLOMORPHIC MAPS

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ABSTRACT. We prove the existence and the essential uniqueness of canonical models for the forward (resp. backward) iteration of a holomorphic self-map f of a cocompact Kobayashi hyperbolic complex manifold, such as the ball $\mathbb{B}^q$ or the polydisc Δ^q. This is done performing a time-dependent conjugacy of the dynamical system (f^n), obtaining in this way a non-autonomous dynamical system admitting a relatively compact forward (resp. backward) orbit, and then proving the existence of a natural complex structure on a suitable quotient of the direct limit (resp. subset of the inverse limit). As a corollary we prove the existence of a holomorphic solution with values in the upper half-plane of the Valiron equation for a holomorphic self-map of the unit ball.

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INTRODUCTION

In order to study the forward or backward iteration of a holomorphic self-map $f\colon X \to X$ of a complex manifold, it is natural to search for a semi-conjugacy of f with some automorphism of a complex manifold. The first example of this approach is old as complex dynamics itself: if $\mathbb{D} \subset \mathbb{C}$

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is the unit disc and $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic self-map such that $f(0) = 0$ and $0 < |f'(0)| < 1$, then in 1884 Königs proved [22] that there exists a unique holomorphic mapping $h\colon \mathbb{D} \to \mathbb{C}$ solving the Schröder equation

$$h \circ f = f'(0)h,$$

and satisfying $h'(0) = 1$. Clearly h gives a semi-conjugacy between f and the automorphism $z \mapsto f'(0)z$ of $\mathbb{C}$. Notice that $\cup_{n\geq 0} f'(0)^{-n}h(\mathbb{D}) = \mathbb{C}$.

We call *semi-model* for f a triple (Λ, h, φ), where Λ is a complex manifold called the *base space*, $h\colon X \to \Lambda$ is a holomorphic mapping called the *intertwining mapping* and $\varphi\colon \Lambda \to \Lambda$ is an automorphism, such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ \Lambda \arrow[r, "\varphi"] & \Lambda, \end{tikzcd}$$

and $\Lambda = \bigcup_{n\geq 0} \varphi^{-n}(h(X))$. A *model* for f is a semi-model (Λ, h, φ) such that the intertwining mapping h is univalent on an f-absorbing domain, that is, a domain A such that $f(A) \subset A$ and such that every orbit of f eventually lies in A.

There is a "dual" way of semi-conjugating f with an automorphism: we call *pre-model* for f a triple (Λ, h, φ), where Λ is a complex manifold called the *base space*, $h\colon \Lambda \to X$ is a holomorphic mapping called the *intertwining mapping* and $\varphi\colon \Lambda \to \Lambda$ is an automorphism, such that the following diagram commutes:

$$\begin{tikzcd} \Lambda \arrow[r, "\varphi"] \arrow[d, "h"'] & \Lambda \arrow[d, "h"] \\ X \arrow[r, "f"] & X. \end{tikzcd}$$

We refer to [4, 3, 2] for a brief history and recent developments in the theories of semi-models and pre-models. We recall that semi-models and pre-models, besides giving informations on the iteration of the self-map f, can also be fruitfully applied to the study of composition operators [7, 15, 21, 24] and of commuting self-maps [14, 8, 12].

We now need to recall some definitions and results for a holomorphic self-map f of the unit disc $\mathbb{D} \subset \mathbb{C}$. A point $\zeta \in \partial\mathbb{D}$ is a *boundary regular fixed point* if $\angle\lim_{z\to\zeta} f(z) = \zeta$, where $\angle\lim$ denotes the non-tangential limit, and if

$$\lambda := \liminf_{z\to\zeta} \frac{1 - |f(z)|}{1 - |z|} < +\infty.$$

The number $\lambda \in (0, +\infty)$ is called the *dilation* of f at ζ. The point ζ is *repelling* if $\lambda > 1$. The classical Denjoy–Wolff theorem states that if f admits no fixed point $z \in \mathbb{D}$, then there exists a boundary regular fixed point $p \in \partial\mathbb{D}$ with dilation $\lambda \leq 1$ such that (f^n) converges to the constant map p uniformly on compact subsets. The self-map f is called *hyperbolic* if $\lambda < 1$. We denote by $\mathbb{H} \subset \mathbb{C}$ the upper half-plane.

We are interested in the following examples of semi-models and pre-models in $\mathbb{D}$, given respectively by Valiron [29] and by Poggi-Corradini [25]. Both examples can be seen as the solution of a generalized Schröder equation at the boundary of the disc.

Theorem 0.1 (Valiron). *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a hyperbolic holomorphic self-map with dilation $\lambda < 1$ at its Denjoy–Wolff point. Then there exists a model $(\mathbb{H}, h, z \mapsto \frac{1}{\lambda} z)$ for f.*

Theorem 0.2 (Poggi-Corradini). *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map and let ζ be a boundary repelling fixed point with dilation $\lambda > 1$. Then there exists a pre-model $(\mathbb{H}, h, z \mapsto \frac{1}{\lambda} z)$ for f.*

A proof of the essential uniqueness of the intertwining mapping in Theorem 0.1 was given by Bracci–Poggi-Corradini [11], and Poggi-Corradini [25] proved that the intertwining mapping in Theorem 0.2 is essentially unique.

These two results were generalized to the unit ball $\mathbb{B}^q \subset \mathbb{C}^q$ (for a definition of dilation, hyperbolic self-maps, Denjoy–Wolff point and boundary repelling points in the ball, see Sections 4 and 8). Bracci–Gentili–Poggi-Corradini [10] studied the case of a hyperbolic holomorphic self-map $f\colon \mathbb{B}^q \to \mathbb{B}^q$ with dilation $\lambda < 1$ at its Denjoy–Wolff point $p \in \partial\mathbb{B}^q$, and, assuming some regularity at p, they proved the existence of a one-dimensional semi-model $(\mathbb{H}, h, z \mapsto \frac{1}{\lambda} z)$ for f (for other results about semi-models for hyperbolic self-maps, see [9, 21, 6]).

Ostapyuk [23] studied the case of a holomorphic self-map $f\colon \mathbb{B}^q \to \mathbb{B}^q$ with a boundary repelling fixed point $\zeta \in \partial\mathbb{B}^q$ with dilation $\lambda > 1$, and, assuming that ζ is isolated from other boundary repelling fixed points with dilation less or equal than λ, she proved the existence of a one-dimensional pre-model $(\mathbb{H}, h, z \mapsto \frac{1}{\lambda} z)$ for f.

Theorems 0.1 and 0.2 were generalized respectively by Bracci and the author [4] and by the author [3] to the case of a univalent self-map $f\colon X \to X$ of a cocompact Kobayashi hyperbolic complex manifold (such as the unit ball $\mathbb{B}^q$ or the unit polydisc Δ^q). The approach used is geometric, much in the spirit of the work of Cowen [13] for the forward iteration in the unit disc.

We first consider the forward iteration case. Let k denote the Kobayashi distance. Notice that if $(z_n := f^n(z_0))$ is a forward orbit, then for all fixed $m \geq 1$ the sequence $(k_X(z_n, z_{n+m}))_{n\geq 0}$ is non-increasing. The limit $s_m(z_0) := \lim_{n\to\infty} k_X(z_n, z_{n+m})$ is called the *forward m-step* at z_0. The *divergence rate* of a self-map is a generalization introduced in [4] of the dilation at the Denjoy–Wolff point of a holomorphic self-map of $\mathbb{B}^q$.

Theorem 0.3 (A.–Bracci). *Let X be Kobayashi hyperbolic and cocompact and let $f\colon X \to X$ be a univalent self-map. Then there exists an essentially unique model (Ω, σ, ψ). Moreover, there exists a holomorphic retract Z of X, a surjective holomorphic submersion $r\colon \Omega \to Z$, and an automorphism $\tau\colon Z \to Z$ with divergence rate*

$$c(\tau) = c(f) = \lim_{m\to\infty} \frac{s_m(x)}{m}, \quad x \in X, \tag{0.1}$$

such that $(Z, r\circ\sigma, \tau)$ is a semi-model for f, called a canonical Kobayashi hyperbolic semi-model.

Moreover, the semi-model $(Z, r\circ\sigma, \tau)$ satisfies the following universal property. If (Λ, h, φ) is a semi-model for f such that Λ is Kobayashi hyperbolic, then there exists a surjective holomorphic mapping $\eta\colon Z \to \Lambda$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "r\circ\sigma"] \arrow[dd, "f"] & & \Lambda \arrow[dd, "\varphi"] \\
& Z \arrow[ur, "\eta"] \arrow[dd, "\tau"] & \\
X \arrow[rr, "h"] \arrow[dr, "r\circ\sigma"] & & \Lambda \\
& Z. \arrow[ur, "\eta"] &
\end{tikzcd}$$

In particular, if $X = \mathbb{B}^q$ and f is hyperbolic with dilation $\lambda < 1$ at its Denjoy–Wolff point, then Z is biholomorphic to a ball $\mathbb{B}^k$ with $1 \leq k \leq q$, and the automorphism τ is hyperbolic with dilation λ at its Denjoy-Wolf point. As a corollary Theorem 0.3 yields the existence of a semi-model $(\mathbb{H}, \vartheta, z \mapsto \frac{1}{\lambda}z)$ for f, hence $\vartheta\colon \mathbb{B}^q \to \mathbb{H}$ is a holomorphic solution of the Valiron equation

$$\vartheta \circ f = \frac{1}{\lambda}\vartheta. \tag{0.2}$$

Now we recall the backward iteration case. A *backward orbit* is a sequence $\beta := (y_n)$ in X such that $f(y_{n+1}) = y_n$ for all $n \geq 0$. Notice that if (y_n) is a backward orbit, then for all fixed $m \geq 1$ the sequence $(k_X(y_n, y_{n+m}))_{n\geq 0}$ is non-decreasing. The limit $\sigma_m(\beta) := \lim_{n\to\infty} k_X(y_n, y_{n+m})$ is called the *backward m-step* of β. A backward orbit β has *bounded step* if $\sigma_1(\beta) < +\infty$.

Theorem 0.4 (A.). *Let X be Kobayashi hyperbolic and cocompact and let $f\colon X \to X$ be a univalent self-map. Let $\beta := (y_n)$ be a backward orbit for f with bounded step. Then there exists a holomorphic retract Z of X, an injective holomorphic immersion $\ell\colon Z \to X$, and an automorphism $\tau\colon Z \to Z$ with divergence rate*

$$c(\tau) = \lim_{m\to\infty} \frac{\sigma_m(\beta)}{m}, \tag{0.3}$$

such that (Z, ℓ, τ) is a pre-model for f, called a canonical pre-model *associated with $[y_n]$.*

Moreover (Z, ℓ, τ) satisfies the following universal property. If (Λ, h, φ) is a pre-model for f such that for some (and hence for any) $w \in \Lambda$, the non-decreasing sequence $(k_X(h(\varphi^{-n}(w)), y_n))_{n\geq 0}$ is bounded, then there exists an injective holomorphic mapping $\eta\colon \Lambda \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
\Lambda \arrow[rr, "h"] \arrow[dr, "\eta"] \arrow[dd, "\varphi"] & & X \arrow[dd, "f"] \\
& Z \arrow[ur, "\ell"] \arrow[dd, "\tau"] & \\
\Lambda \arrow[rr, "h"] \arrow[dr, "\eta"] & & X \\
& Z. \arrow[ur, "\ell"] &
\end{tikzcd}$$

In particular, if $X = \mathbb{B}^q$ and the backward orbit (y_n) converges to a boundary repelling fixed point $\zeta \in \partial\mathbb{B}^q$ with dilation $\lambda > 1$, then Z is biholomorphic to a ball $\mathbb{B}^k$ with $1 \leq k \leq q$, and the automorphism τ is hyperbolic with dilation $\mu \geq \lambda$ at its unique boundary repelling fixed point.

In this paper we generalize Theorems 0.3 and 0.4 to non-necessarily univalent holomorphic self-maps $f\colon X \to X$, and then we apply our results to the case of the unit ball $\mathbb{B}^q$. Our proofs underline the strong duality between the forward case and the backward case.

In the first part of the paper we prove Theorem 3.6, which generalizes Theorem 0.3. Let $(\Omega, \Lambda_n\colon X \to \Omega)$ be the direct limit of the sequence $(f^n\colon X \to X)$. Consider the equivalence relation $\sim$ on Ω, where $[(x,n)], [(y,u)] \in \Omega$ are equivalent by $\sim$ if and only if

$$k_X(f^{m-n}(x), f^{m-u}(y)) \overset{m\to\infty}{\longrightarrow} 0.$$

The bijective self-map $\psi\colon \Omega \to \Omega$ defined by $[(x,n)] \mapsto [(f(x),n)]$ satisfies $\psi \circ \Lambda_0 = \Lambda_0 \circ f$ and passes to the quotient inducing a bijective self-map $\hat\psi\colon \Omega/_\sim \to \Omega/_\sim$ satisfying

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\hat\Lambda_0"'] & X \arrow[d, "\hat\Lambda_0"] \\
\Omega/_\sim \arrow[r, "\hat\psi"] & \Omega/_\sim,
\end{tikzcd}$$

where $\hat\Lambda_0 := \pi_\sim \circ \Lambda_0$. A natural candidate for a canonical Kobayashi hyperbolic semi-model for f would be the triple $(\Omega/_\sim, \Lambda_0, \hat\psi)$. Indeed, by the universal property of the direct limit, if (Λ, h, φ) is a semi-model for f such that Λ is Kobayashi hyperbolic, then there exists a mapping $\eta\colon \Omega/_\sim \to \Lambda$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\hat\Lambda_0"] \arrow[dd, "f"] & & \Lambda \arrow[dd, "\varphi"] \\
& \Omega/_\sim \arrow[ur, "\eta"] \arrow[dd, "\hat\psi"] & \\
X \arrow[rr, "h"] \arrow[dr, "\hat\Lambda_0"] & & \Lambda \\
& \Omega/_\sim. \arrow[ur, "\eta"] &
\end{tikzcd}$$

We have to show that $\Omega/_\sim$ can be endowed with a suitable complex structure. If f is univalent, then it follows from the proof of Theorem 0.3 that the direct limit Ω admits a natural complex structure which passes to the quotient to a complex structure on $\Omega/_\sim$ (see [4]). The problem in the non-univalent case is that Ω may not admit a natural complex structure. Rather surprisingly, even if Ω does not, the quotient set $\Omega/_\sim$ can always be endowed with a complex structure which makes it biholomorphic to a holomorphic retract of X. We prove this by conjugating (f^n) to a non-autonomous holomorphic forward dynamical system $(\tilde f_{n,m}\colon X \to X)_{m\geq n\geq 0}$ which admits a relatively compact forward orbit. This orbit is used to prove the existence of a holomorphic

retract Z of X and a family of holomorphic mappings $(\alpha_n\colon X \to Z)$ satisfying

$$\alpha_m \circ \tilde{f}_{n,m} = \alpha_n, \quad \forall\, 0 \leq n \leq m.$$

By the universal property of the direct limit there exists a mapping $\Phi\colon \Omega \to Z$ which induces a bijection $\hat{\Phi}\colon \Omega/_{\sim} \to Z$, which pulls back the desired complex structure to $\Omega/_{\sim}$. Formula (0.1) for the divergence rate of τ is a consequence of the fact that the Kobayashi distance on $\Omega/_{\sim}$ admits a description in terms of the forward iteration of f.

In the second part of the paper, we consider the backward iteration of $f\colon X \to X$ and we prove Theorem 7.5, which generalizes Theorem 0.4. Let $(\Theta, V_n\colon \Theta \to X)$ be the inverse limit of the sequence $(f^n\colon X \to X)$. Let (y_n) be a backward orbit with bounded step and let $[y_n] \subset \Theta$ be the subset consisting of the backward orbits $(z_n) \in \Theta$ such that the non-decreasing sequence $(k_X(z_n, y_n))_{n\geq 0}$ is bounded. The bijective self-map $\psi\colon \Theta \to \Theta$ defined by $(z_0, z_1, z_2, \dots) \mapsto [(f(z_0), z_0, z_1, \dots)]$ satisfies $\psi([y_n]) = [y_n]$, and the following diagram commutes:

$$\begin{tikzcd}
{[y_n]} \arrow[r, "\psi|_{[y_n]}"] \arrow[d, "V_0"'] & {[y_n]} \arrow[d, "V_0"] \\
X \arrow[r, "f"] & X.
\end{tikzcd}$$

A natural candidate for a canonical pre-model for f associated with $[y_n]$ would be the triple $([y_n], V_0, \psi|_{[y_n]})$. Indeed, by the universal property of the inverse limit, if (Λ, h, φ) is a pre-model for f such that for some (and hence for any) $w \in \Lambda$ the non-decreasing sequence $(k_X(h(\varphi^{-n}(w)), y_n))_{n\geq 0}$ is bounded, then there exists a mapping $\eta\colon \Lambda \to [y_n]$ which makes the following diagram commute:

$$\begin{tikzcd}
\Lambda \arrow[rr, "h"] \arrow[dr, "\eta"] \arrow[dd, "\varphi"'] & & X \arrow[dd, "f"] \\
& {[y_n]} \arrow[ur, "V_0"] \arrow[dd, "\psi|_{[y_n]}"] & \\
\Lambda \arrow[rr, "h"] \arrow[dr, "\eta"] & & X \\
& {[y_n].} \arrow[ur, "V_0"] &
\end{tikzcd}$$

We have to show that $[y_n]$ can be endowed with a suitable complex structure. If f is univalent, then $V_0\colon \Theta \to X$ is injective, and it follows from the proof of Theorem 0.4 that the image $V_0([y_n])$ is an injectively immersed complex submanifold of X which is biholomorphic to a holomorphic retract of X. In the non-univalent case the mapping $V_0\colon \Theta \to X$ is no longer injective, but the subset $[y_n]$ can however be endowed with a natural complex structure which makes it biholomorphic to a holomorphic retract of X. We prove this by conjugating (f^n) to a non-autonomous holomorphic backward dynamical system $(\tilde{f}_{n,m}\colon X \to X)_{m\geq n\geq 0}$ which admits a relatively compact backward orbit. This orbit is used to prove the existence of a holomorphic

retract Z of X and a family of holomorphic mappings $(\alpha_n\colon Z \to X)$ satisfying

$$\tilde{f}_{n,m} \circ \alpha_m = \alpha_n, \quad \forall\ 0 \le n \le m.$$

By the universal property of the inverse limit there exists an injective mapping $\Phi\colon Z \to \Theta$, which pushes forward the desired complex structure to its image $\Phi(Z) = [y_n]$. Formula (0.3) for the divergence rate of τ is a consequence of the fact that the Kobayashi distance of $[y_n]$ admits a description in terms of the backward iteration of f.

Part 1. Forward iteration

1. Preliminaries

Definition 1.1. Let X be a complex manifold. We call *forward (non-autonomous) holomorphic dynamical system* on X any family $(f_{n,m}\colon X \to X)_{m \ge n \ge 0}$ of holomorphic self-maps such that for all $m \ge u \ge n \ge 0$, we have

$$f_{u,m} \circ f_{n,u} = f_{n,m}.$$

For all $n \ge 0$ we denote $f_{n,n+1}$ also by f_n. A forward holomorphic dynamical system $(f_{n,m}\colon X \to X)_{m \ge n \ge 0}$ is called *autonomous* if $f_n = f_0$ for all $n \ge 0$. Clearly in this case $f_{n,m} = f_0^{m-n}$.

Remark 1.2. Any family of holomorphic self-maps $(f_n\colon X \to X)_{n \ge 0}$ determines a forward holomorphic dynamical system $(f_{n,m}\colon X \to X)$ in the following way: for all $n \ge 0$, set $f_{n,n} = \mathsf{id}$, and for all $m > n \ge 0$, set

$$f_{n,m} = f_{m-1} \circ \cdots \circ f_n.$$

Definition 1.3. Let X be a complex manifold, and let $(f_{n,m}\colon X \to X)$ be a forward holomorphic dynamical system. A *direct limit* for $(f_{n,m})$ is a pair (Ω, Λ_n) where Ω is a set and $(\Lambda_n\colon X \to \Omega)_{n \ge 0}$ is a family of mappings such that

$$\Lambda_m \circ f_{n,m} = \Lambda_n, \quad \forall\ m \ge n \ge 0,$$

satisfying the following universal property: if Q is a set and if $(g_n\colon X \to Q)$ is a family of mappings satisfying

$$g_m \circ f_{n,m} = g_n, \quad \forall\ m \ge n \ge 0,$$

then there exists a unique mapping $\Gamma\colon \Omega \to Q$ such that

$$g_n = \Gamma \circ \Lambda_n, \quad \forall\ n \ge 0.$$

Remark 1.4. The direct limit is essentially unique, in the following sense. Let (Ω, Λ_n) and (Q, g_n) be two direct limits for $(f_{n,m})$. Then there exists a bijective mapping $\Gamma\colon \Omega \to Q$ such that

$$g_n = \Gamma \circ \Lambda_n, \quad \forall\ n \ge 0.$$

Remark 1.5. A direct limit for $(f_{n,m})$ is easily constructed. We define an equivalence relation on the set $X \times \mathbb{N}$ in the following way: $(x, n) \simeq (y, m)$ if and only if there exists $u \ge \max\{n, m\}$ such that $f_{n,u}(x) = f_{m,u}(y)$. We denote the equivalence class of (x, n) by $[(x, n)]$, and we set $\Omega \coloneqq X \times \mathbb{N}/_{\simeq}$. We define a family of mappings $(\Lambda_n\colon X \to \Omega)_{n \ge 0}$ in the following way: for all $x \in X$ and $n \ge 0$, set $\Lambda_n(x) = [(x, n)]$. It is easy to see that (Ω, Λ_n) is a direct limit for $(f_{n,m})$.

Definition 1.6. In what follows we will need the following equivalence relation on Ω:

$$[(x,n)] \sim [(y,u)] \quad \text{iff} \quad k_X(f_{n,m}(x), f_{u,m}(y)) \overset{m\to\infty}{\longrightarrow} 0.$$

It is easy to see that this is well-defined. We denote by $\pi_\sim\colon \Omega \to \Omega/_\sim$ the projection to the quotient.

We now introduce a modified version of the direct limit for $(f_{n,m})$ which is more suited for our needs.

Definition 1.7. Let X be a complex manifold and let $(f_{n,m}\colon X \to X)$ be a forward holomorphic dynamical system. We call *canonical Kobayashi hyperbolic direct limit* for $(f_{n,m})$ a pair (Z, α_n) where Z is a Kobayashi hyperbolic complex manifold and $(\alpha_n\colon X \to Z)_{n\geq 0}$ is a family of holomorphic mappings such that

$$\alpha_m \circ f_{n,m} = \alpha_n, \quad \forall\, m \geq n \geq 0,$$

which satisfies the following universal property: if Q is a Kobayashi hyperbolic complex manifold and if $(g_n\colon X \to Q)$ is a family of holomorphic mappings satisfying

$$g_m \circ f_{n,m} = g_n, \quad \forall\, m \geq n \geq 0,$$

then there exists a unique holomorphic mapping $\Gamma\colon Z \to Q$ such that

$$g_n = \Gamma \circ \alpha_n, \quad \forall\, n \geq 0.$$

Proposition 1.8. *The canonical Kobayashi hyperbolic direct limit is essentially unique, in the following sense. Let (Z, α_n) and (Q, g_n) be two canonical Kobayashi hyperbolic direct limits for $(f_{n,m})$. Then there exists a biholomorphism $\Gamma\colon Z \to Q$ such that*

$$g_n = \Gamma \circ \alpha_n, \quad \forall\, n \geq 0.$$

Proof. There exist holomorphic mappings $\Gamma\colon Z \to Q$ and $\Xi\colon Q \to Z$ such that for all $n \geq 0$, we have $g_n = \Gamma \circ \alpha_n$ and $\alpha_n = \Xi \circ g_n$. Thus the holomorphic mapping $\Xi \circ \Gamma\colon Z \to Z$ satisfies

$$\Xi \circ \Gamma \circ \alpha_n = \alpha_n, \quad \forall\, n \geq 0,$$

By the universal property of the canonical Kobayashi hyperbolic direct limit, this implies that $\Xi \circ \Gamma = \mathsf{id}_Z$. Similarly, we obtain $\Gamma \circ \Xi = \mathsf{id}_Q$. $\square$

2. Non-autonomous iteration

Let X be a taut complex manifold. Let $(f_{n,m}\colon X \to X)_{m\geq n\geq 0}$ be a forward holomorphic dynamical system, and assume that it admits a relatively compact forward orbit $(f_{0,m}(x_0))_{m\geq 0}$.

Remark 2.1. Let $K \subset X$ be a compact subset such that $\{f_{0,m}(x_0)\}_{m\geq 0} \subset K$. It follows that, for all fixed $n \geq 0$,

$$f_{n,m}(K) \cap K \neq \varnothing \quad \forall\, m \geq n. \tag{2.1}$$

The sequence of holomorphic self-maps $(f_{0,m}\colon X \to X)_{m\geq 0}$ is not compactly divergent by (2.1), and since X is taut, there exists a subsequence $(f_{0,m_{k_0}})_{k_0\geq 0}$ converging uniformly on compact subsets to a holomorphic self-map $\alpha_0\colon X \to X$. The sequence of holomorphic self-maps $(f_{1,m_{k_0}}\colon X \to X)_{k_0\geq 0}$ is not compactly divergent by (2.1), and since X is taut, there exists a subsequence $(f_{1,m_{k_1}})_{k_1\geq 0}$ converging to a holomorphic self-map $\alpha_1\colon X \to X$. Iterating

this procedure we obtain a family of holomorphic self-maps $(\alpha_n\colon X \to X)$ satisfying for all $m \geq n \geq 0$,

$$\alpha_m \circ f_{n,m} = \alpha_n. \tag{2.2}$$

Notice that for all $n \geq 0$ we have

$$\alpha_n(K) \cap K \neq \varnothing. \tag{2.3}$$

Let now $(m_k)_{k\geq 0}$ be a sequence of integers which for all $j \geq 0$ is eventually a subsequence of $(m_{k_j})_{k_j\geq 0}$ (such a sequence exists by a classical diagonal argument).

The sequence of holomorphic self-maps $(\alpha_{m_k}\colon X \to X)_{k\geq 0}$ is not compactly divergent by (2.3), and since X is taut, there exists a subsequence $(\alpha_{m_h})_{h\geq 0}$ converging uniformly on compact subsets to a holomorphic self-map $\alpha\colon X \to X$.

Proposition 2.2. *The holomorphic self-map $\alpha\colon X \to X$ is a holomorphic retraction, and for all $n \geq 0$,*

$$\alpha \circ \alpha_n = \alpha_n. \tag{2.4}$$

Proof. Fix $n \geq 0$ and $x \in X$. Then for all $h \geq 0$ such that $m_h \geq n$, we have

$$\alpha_n(x) = \alpha_{m_h}(f_{n,m_h}(x)) \overset{h\to\infty}{\longrightarrow} \alpha(\alpha_n(x)).$$

Thus we have, for all $h \geq 0$,

$$\alpha(\alpha_{m_h}(x)) = \alpha_{m_h}(x).$$

When $h \to \infty$, the left-hand side converges to $\alpha(\alpha(x))$, while the right-hand side converges to $\alpha(x)$. □

Remark 2.3. The image $\alpha(X)$ is a closed complex submanifold of X (see e.g. [1, Lemma 2.1.28]).

Definition 2.4. We denote $\alpha(X)$ by Z.

Remark 2.5. By (2.4), we have $\alpha_n(X) \subset Z$ for all $n \geq 0$, and by (6.2) we have that

$$\alpha_n(X) \subset \alpha_m(X)$$

for all $m \geq n \geq 0$.

Let (Ω, Λ_n) be the direct limit of the directed system $(X, f_{n,m})$. By the universal property of the direct limit, there exists a mapping $\Psi\colon \Omega \to Z$ such that for all $n \geq 0$,

$$\alpha_n = \Psi \circ \Lambda_n.$$

The mapping Ψ is defined in the following way: if $[(x,n)] \in \Omega$, then $\Psi([(x,n)]) = \alpha_n(x)$.

Proposition 2.6. *The mapping $\Psi\colon \Omega \to Z$ is surjective, and $\Psi([(x,n)]) = \Psi([(y,u)])$ if and only if $[(x,n)] \sim [(y,u)]$.*

Proof. Since α is a retraction, we have $\alpha(z) = z$ for all $z \in Z$, that is, $\alpha_{m_h}(z) \overset{h\to\infty}{\longrightarrow} z$ for all $z \in Z$. Consider the sequence of holomorphic mappings $(\alpha_{m_h}|_Z\colon Z \to Z)$. This sequence converges uniformly on compact subsets to id_Z, and thus it is eventually injective on compact subsets of Z. Fix $z \in Z$ and let U be a neighborhood of z in Z such that $(\alpha_{m_h}|_U\colon U \to Z)$ is eventually injective. Then the image $\alpha_{m_h}|_U$ eventually contains z (see e.g. [5, Corollary 3.2]). Hence we obtain that $\Psi\colon \Omega \to Z$ is surjective.

Assume now that $[(x,n)] \sim [(y,u)]$. For all $m \geq \max\{n,u\}$, we have that $\Psi([(x,n)]) = \alpha_m(f_{n,m}(x))$, and $\Psi([(y,u)]) = \alpha_m(f_{u,m}(y))$. We have

$$k_X(\Psi([(x,n)]), \Psi([(y,u)])) \leq k_X(f_{n,m}(x), f_{u,m}(y)) \stackrel{m\to\infty}{\longrightarrow} 0,$$

which implies $\Psi([(x,n)]) = \Psi([(y,u)])$.

Conversely, assume that $\Psi([(x,n)]) = \Psi([(y,u)])$. It follows that

$$\lim_{h\to\infty} f_{n,m_h}(x) = \lim_{h\to\infty} f_{u,m_h}(y),$$

and thus $\lim_{h\to\infty} k_X(f_{n,m_h}(x), f_{u,m_h}(y)) = 0$. Since the sequence $(k_X(f_{n,m}(x), f_{u,m}(y)))_{m\geq \max\{n,u\}}$ is non-increasing, we have $[(x,n)] \sim [(y,u)]$. $\square$

Remark 2.7. It follows from Proposition 2.6 that $\bigcup_{n\geq 0} \alpha_n(X) = Z$, and that Ψ induces a bijection $\hat{\Psi}\colon \Omega/_\sim \to Z$.

Proposition 2.8. *The pair (Z, α_n) is a canonical Kobayashi hyperbolic direct limit for $(f_{n,m})$.*

Proof. First of all, Z is Kobayashi hyperbolic since it is a submanifold of X. Let Q be a Kobayashi hyperbolic complex manifold and let $(g_n\colon X \to Q)$ be a family of holomorphic mappings satisfying

$$g_m \circ f_{n,m} = g_n, \quad \forall\, m \geq n \geq 0.$$

By the universal property of the direct limit, there exists a unique mapping $\Phi\colon \Omega \to Q$ such that

$$g_n = \Phi \circ \Lambda_n, \quad \forall\, n \geq 0.$$

The mapping Φ is defined in the following way: if $[(x,n)] \in \Omega$, then $\Phi([(x,n)]) = g_n(x)$. We claim that

$$[(x,n)] \sim [(y,u)] \Longrightarrow \Phi([(x,n)]) = \Phi[(y,u)].$$

Indeed, if $[(x,n)] \sim [(y,u)]$, then for all $m \geq \max\{n,u\}$, we have that $\Phi([(x,n)]) = g_m(f_{n,m}(x))$, and $\Phi([(y,u)]) = g_m(f_{u,m}(y))$. We have

$$k_X(\Phi([(x,n)]), \Phi([(y,u)])) \leq k_X(f_{n,m}(x), f_{u,m}(y)) \stackrel{m\to\infty}{\longrightarrow} 0.$$

Thus there exists a unique mapping $\hat{\Phi}\colon \Omega/_\sim \to Q$ such that $\hat{\Phi} \circ \pi_\sim = \Phi$.

Set

$$\Gamma := \hat{\Phi} \circ \hat{\Psi}^{-1}\colon Z \to Q.$$

For all $n \geq 0$,

$$\Gamma \circ \alpha_n = \Gamma \circ \Psi \circ \Lambda_n = \hat{\Phi} \circ \pi_\sim \circ \Lambda_n = \Phi \circ \Lambda_n = g_n.$$

The uniqueness of the mapping Γ follows easily from the uniqueness of the mappings Φ and $\hat{\Phi}$. The mapping Γ acts in the following way: if $z \in Z$, then there exists $x \in X$ and $n \geq 0$ such that $\alpha_n(x) = z$, and then $\Gamma(z) = g_n(x)$.

We now prove that Γ is holomorphic. Let $z \in Z$, and let $x \in X$ and $n \geq 0$ such that $\alpha_n(x) = z$. Since α has maximal rank at z, there exists a neighborhood V of z in X such that, for m large enough, α_m has maximal rank at every point $y \in V$. Since the sequence $(f_{n,m_{k_n}}(x))_{k_n\geq 0}$ converges to $\alpha_n(x) = z$ as $k_n \to \infty$, it is eventually contained in V. Hence there

exists $m' \geq 0$ such that $w := f_{n,m'}(x) \in V$ and $\alpha_{m'}$ has maximal rank at w. Thus there exists an open neighborhood $U \subset Z$ of z and a holomorphic function $\sigma\colon U \to X$ such that

$$\alpha_{m'} \circ \sigma = \mathsf{id}_U.$$

Then, for all $y \in U$,

$$\Gamma(y) = \Gamma(\alpha_{m'}(\sigma(y))) = g_{m'}(\sigma(y)),$$

which means that Γ is holomorphic in U. □

We denote by κ the Kobayashi–Royden metric.

Proposition 2.9. *For all $n \geq 0$,*

$$\lim_{m\to\infty} f^*_{n,m}\, k_X = \alpha^*_n\, k_Z, \tag{2.5}$$

and

$$\lim_{m\to\infty} f^*_{n,m}\, \kappa_X = \alpha^*_n\, \kappa_Z. \tag{2.6}$$

Proof. Let $x, y \in X$, and fix $n \geq 0$. We have that

$$\lim_{k_n\to\infty} k_X(f_{n,m_{k_n}}(x), f_{n,m_{k_n}}(y)) = k_X(\alpha_n(x), \alpha_n(y)) = k_Z(\alpha_n(x), \alpha_n(y)),$$

where the last identity follows from the fact that $\alpha_n(x), \alpha_n(y) \in Z$ and Z is a holomorphic retract. Then (2.5) follows since the sequence $(k_X(f_{n,m}(x), f_{n,m}(y)))_{m\geq n}$ is non-increasing.

The proof of (2.6) is similar. □

Definition 2.10. Let X be a Kobayashi hyperbolic complex manifold. We say that X is *cocompact* if $X/\mathrm{aut}(X)$ is compact.

Notice that this implies that X is complete Kobayashi hyperbolic [18, Lemma 2.1].

Theorem 2.11. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $(f_{n,m}\colon X \to X)_{m\geq n\geq 0}$ be a forward holomorphic dynamical system. Then there exists a canonical Kobayashi hyperbolic direct limit (Z, α_n) for $(f_{n,m})$, where Z is a holomorphic retract of X. Moreover,*

$$Z = \bigcup_{n\geq 0} \alpha_n(X), \tag{2.7}$$

and

$$\lim_{m\to\infty} f^*_{n,m}\, k_X = \alpha^*_n\, k_Z, \quad \lim_{m\to\infty} f^*_{n,m}\, \kappa_X = \alpha^*_n\, \kappa_Z. \tag{2.8}$$

Proof. Let $K \subset X$ be a compact subset such that $X = \mathrm{Aut}(X) \cdot K$. Let $x_0 \in X$. For all $n \geq 0$, let $h_n \in \mathrm{Aut}(X)$ be such that $h_n(f_{0,n}(x_0)) \in K$. For all $m \geq n \geq 0$ set $\tilde{f}_{n,m} := h_m \circ f_{n,m} \circ h_n^{-1}$. It is easy to see that $(\tilde{f}_{n,m}\colon X \to X)$ is a forward holomorphic dynamical system such that

$$\{\tilde{f}_{0,m}(h_0(x_0))\}_{m\geq 0} \subset K. \tag{2.9}$$

We can now apply Proposition 2.8 to $(\tilde{f}_{n,m}\colon X \to X)$, obtaining a canonical Kobayashi hyperbolic direct limit $(Z, \tilde{\alpha}_n)$ for $(\tilde{f}_{n,m})$, where Z is a holomorphic retract of X. For all $n \geq 0$ set $\alpha_n := \tilde{\alpha}_n \circ h_n$. Clearly

$$\alpha_m \circ f_{n,m} = \alpha_n, \quad \forall\, m \geq n \geq 0.$$

Let Q be a Kobayashi hyperbolic manifold and let $(g_n\colon X \to Q)$ be a family of holomorphic mappings satisfying

$$g_m \circ f_{n,m} = g_n, \quad \forall\, m \geq n \geq 0.$$

For all $n \geq 0$ set $\tilde{g}_n \coloneqq g_n \circ h_n^{-1}$. Then for all $m \geq n \geq 0$,

$$\tilde{g}_m \circ \tilde{f}_{n,m} = g_m \circ h_m^{-1} \circ \tilde{f}_{n,m} = g_m \circ f_{n,m} \circ h_n^{-1} = g_n \circ h_n^{-1} = \tilde{g}_n.$$

By the universal property of the canonical Kobayashi hyperbolic direct limit applied to $(Z, \tilde{\alpha}_n)$ we obtain a holomorphic mapping $\Gamma\colon Z \to Q$ such that

$$\tilde{g}_n = \Gamma \circ \tilde{\alpha}_n, \quad \forall\, n \geq 0.$$

Hence $g_n = \Gamma \circ \alpha_n$ for all $n \geq 0$.

Remark 2.7 yields (2.7). Finally, (2.8) follows from Proposition 2.9 since for all $n \geq 0$ the automorphism $h_n\colon X \to X$ is an isometry for k_X and κ_X. $\square$

Remark 2.12. Let (Ω, Λ_n) be the direct limit of the directed system $(X, f_{n,m})$. Let (Z, α_n) be the canonical Kobayashi hyperbolic direct limit given by Theorem 2.11. By the universal property of the direct limit, there exists a mapping $\Psi\colon \Omega \to Z$ such that $\alpha_n = \Psi \circ \Lambda_n$ for all $n \geq 0$. It is easy to see that Ψ is surjective and induces a bijection $\hat{\Psi}\colon \Omega/_\sim \to Z$ such that

$$\alpha_n = \hat{\Psi} \circ \pi_\sim \circ \Lambda_n, \quad \forall\, n \geq 0.$$

3. Autonomous iteration

Definition 3.1. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. Let $x \in X$, and let $m \geq 0$. The *m-step* $s_m(x)$ of f at x is the limit

$$s_m(x) = \lim_{n\to\infty} k_X(f^n(x), f^{n+m}(x)).$$

Such a limit exists since the sequence $(k_X(f^n(x), f^{n+m}(x))_{n\geq 0}$ is non-increasing. The *divergence rate* $c(f)$ of f is the limit

$$c(f) = \lim_{m\to\infty} \frac{k_X(f^m(x), x)}{m}.$$

It is shown in [4] that such a limit exists, does not depend on $x \in X$ and equals $\inf_{m\in\mathbb{N}} \frac{k_X(f^m(x),x)}{m}$.

Definition 3.2. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. A *semi-model* for f is a triple (Λ, h, φ) where Λ is a complex manifold, $h\colon X \to \Lambda$ is a holomorphic mapping, and $\varphi\colon \Omega \to \Omega$ is an automorphism such that

$$h \circ f = \varphi \circ h, \tag{3.1}$$

and

$$\bigcup_{n\geq 0} \varphi^{-n}(h(X)) = \Lambda. \tag{3.2}$$

We call the manifold Λ the *base space* and the mapping h the *intertwining mapping*.

Let (Z, ℓ, τ) and (Λ, h, φ) be two semi-models for f. A *morphism of semi-models* $\hat{\eta}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$ is given by a holomorphic map $\eta : Z \to \Lambda$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\ell"] \arrow[dd, "f"'] & & \Lambda \arrow[dd, "\varphi"] \\
& Z \arrow[ur, "\eta"] \arrow[dd, "\tau"] & \\
X \arrow[rr, "h"] \arrow[dr, "\ell"'] & & \Lambda \\
& Z. \arrow[ur, "\eta"'] &
\end{tikzcd}$$

If the mapping $\eta\colon Z \to \Lambda$ is a biholomorphism, then we say that $\hat{\eta}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$ is an *isomorphism of semi-models*. Notice that then $\eta^{-1}\colon \Lambda \to Z$ induces a morphism $\hat{\eta}^{-1}\colon (\Lambda, h, \varphi) \to (Z, \ell, \tau)$.

Remark 3.3. It is shown in [4, Lemmas 3.6 and 3.7] that if $(Z, \ell, \tau), (\Lambda, h, \varphi)$ are semi-models for f, then there exists at most one morphism $\hat{\eta}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$, and that the holomorphic map $\eta : Z \to \Lambda$ is surjective.

Definition 3.4. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. Let (Z, ℓ, τ) be a semi-model for f whose base space Z is Kobayashi hyperbolic. We say that (Z, ℓ, τ) is a *canonical Kobayashi hyperbolic semi-model* for f if for any semi-model (Λ, h, φ) for f such that the base space Λ is Kobayashi hyperbolic, there exists a morphism of semi-models $\hat{\eta}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$ (which is necessarily unique by Remark 3.3).

Remark 3.5. If (Z, ℓ, τ) and (Λ, h, φ) are two canonical Kobayashi hyperbolic semi-models for f, then they are isomorphic.

Theorem 3.6. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $f\colon X \to X$ be a holomorphic self-map. Then there exists a canonical Kobayashi hyperbolic semi-model (Z, ℓ, τ) for f, where Z is a holomorphic retract of X. Moreover, the following holds:*

(1) *if $\alpha_n := \tau^{-n} \circ \ell$ for all $n \geq 0$, then*

$$\lim_{m\to\infty} (f^m)^* \, k_X = \alpha_n^* \, k_Z, \quad \lim_{m\to\infty} (f^m)^* \, \kappa_X = \alpha_n^* \, \kappa_Z,$$

(2) *the divergence rate of τ satisfies*

$$c(\tau) = c(f) = \lim_{m\to\infty} \frac{s_m(x)}{m} = \inf_{m\in\mathbb{N}} \frac{s_m(x)}{m}.$$

Proof. Let $(f_{n,m}\colon X \to X)$ be the autonomous dynamical system defined by $f_{n,m} = f^{m-n}$. By Theorem 2.11, there exist a holomorphic retract Z of X and a family of holomorphic mappings $(\alpha_n\colon X \to Z)$ such that the pair (Z, α_n) is a canonical Kobayashi hyperbolic direct limit for $(f_{n,m})$. The sequence of holomorphic mappings $(\beta_n := \alpha_n \circ f\colon X \to Z)$ satisfies, for all $m \geq n \geq 0$,

$$\beta_m \circ f_{n,m} = \alpha_m \circ f \circ f^{m-n} = \alpha_n \circ f = \beta_n.$$

By the universal property of the canonical Kobayashi hyperbolic direct limit there exists a holomorphic self-map $\tau\colon Z \to Z$ such that for all $n \geq 0$,

$$\tau \circ \alpha_n = \alpha_n \circ f.$$

We claim that τ is a holomorphic automorphism. For all $n \geq 0$, set $\gamma_n := \alpha_{n+1}$. For all $m \geq n \geq 0$,

$$\gamma_m \circ f_{n,m} = \alpha_{m+1} \circ f^{m-n} = \alpha_{n+1} = \gamma_n.$$

Thus there exists a holomorphic self-map $\delta\colon Z \to Z$ such that $\delta \circ \alpha_n = \alpha_{n+1}$ for all $n \geq 0$. For all $n \geq 0$ we have

$$\tau \circ \delta \circ \alpha_n = \tau \circ \alpha_{n+1} = \alpha_n,$$

and

$$\delta \circ \tau \circ \alpha_n = \delta \circ \alpha_n \circ f = \alpha_{n+1} \circ f = \alpha_n.$$

By the universal property of the canonical Kobayashi hyperbolic direct limit we have that τ is a holomorphic automorphism and $\delta = \tau^{-1}$. Since for all $n \geq 0$,

$$\tau^n \circ \alpha_n = \alpha_n \circ f^n = \alpha_0,$$

it follows that $\alpha_n = \tau^{-n} \circ \alpha_0$.

Set $\ell := \alpha_0$. We claim that the triple (Z, ℓ, τ) is a canonical Kobayashi hyperbolic semi-model for f. Indeed, let (Λ, h, φ) be a semi-model for f such that the base space Λ is Kobayashi hyperbolic. For all $n \geq 0$, let $\lambda_n := \varphi^{-n} \circ h$. Then by the universal property of the canonical Kobayashi hyperbolic direct limit there exists a holomorphic mapping $\eta\colon Z \to \Lambda$ such that for all $n \geq 0$ we have $\eta \circ \alpha_n = \lambda_n$, that is

$$\eta \circ \tau^{-n} \circ \ell = \varphi^{-n} \circ h.$$

Notice that this implies $\eta \circ \ell = h$, and if $n \geq 0$,

$$\varphi \circ \eta \circ \tau^{-1} \circ \alpha_n = \varphi \circ \varphi^{-n-1} \circ h = \lambda_n = \eta \circ \alpha_n.$$

Thus by the universal property of the canonical Kobayashi hyperbolic direct limit, $\eta = \varphi \circ \eta \circ \tau^{-1}$. Hence the mapping $\eta\colon Z \to \Lambda$ gives a morphism of semi-models $\hat{\eta}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$.

Property (1) follows clearly from Theorem 2.11. Property (1) implies in particular that for all $m \geq 0$ and $x \in X$, the m-step $s_m(x)$ satisfies

$$s_m(x) = k_Z(\ell(z), \tau^m(\ell(z))).$$

By [4, Proposition 2.7]

$$c(\tau) = \lim_{m\to\infty} \frac{k_Z(\ell(z), \tau^m(\ell(z)))}{m} = \lim_{m\to\infty} \frac{s_m(x)}{m} = \lim_{m\to\infty} \frac{k_X(f^m(x), x)}{m} = c(f),$$

and

$$c(\tau) = \inf_{m\in\mathbb{N}} \frac{k_Z(\ell(z), \tau^m(\ell(z)))}{m} = \inf_{m\in\mathbb{N}} \frac{s_m(x)}{m},$$

which proves Property (2).

$\square$

Remark 3.7. Actually, the proof shows that the semi-model (Z, ℓ, τ) satisfies the following stronger universal property. If Λ is a Kobayashi hyperbolic complex manifold, if $\varphi\colon \Lambda \to \Lambda$ is an automorphism and if $h\colon X \to \Lambda$ is a holomorphic mapping such that $h \circ f = \varphi \circ h$ (notice that we do not assume (3.2)), then there exists a holomorphic mapping $\eta\colon Z \to \Lambda$ such that $\eta \circ \ell = h$ and $\eta \circ \tau = \varphi \circ \eta$. Clearly, $\eta(Z) = \cup_{n \geq 0} \varphi^{-n} h(X)$.

4. The unit ball

Definition 4.1. The Siegel upper half-space $\mathbb{H}^q$ is defined by

$$\mathbb{H}^q = \left\{(z, w) \in \mathbb{C} \times \mathbb{C}^{q-1}, \mathsf{Im}\,(z) > \|w\|^2\right\}.$$

Recall that $\mathbb{H}^q$ is biholomorphic to the ball $\mathbb{B}^q$ via the *Cayley transform* $\Psi\colon \mathbb{B}^q \to \mathbb{H}^q$ defined as

$$\Psi(z, w) = \left(i\frac{1+z}{1-z}, \frac{w}{1-z}\right), \quad (z, w) \in \mathbb{C} \times \mathbb{C}^{q-1}.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian product in $\mathbb{C}^q$. In several complex variables, the natural generalization of the non-tangential limit at the boundary is the following. If $\zeta \in \partial\mathbb{B}^q$, then the set

$$K(\zeta, R) := \{z \in \mathbb{B}^q : |1 - \langle z, \zeta \rangle| < R(1 - \|z\|)\}$$

is a *Korányi region* of *vertex* ζ and *amplitude* $R > 1$. Let $f\colon \mathbb{B}^q \to \mathbb{C}^m$ be a holomorphic map. We say that f has *K-limit* $L \in \mathbb{C}^m$ at ζ (we write $K\text{-}\lim_{z \to \zeta} f(z) = L$) if for each sequence (z_n) converging to ζ such that (z_n) belongs eventually to some Korányi region of vertex ζ, we have that $f(z_n) \to L$. The Korányi regions can also be easily described in the Siegel upper half-space $\mathbb{H}^q$, see e.g. [10].

Let $\zeta \in \partial\mathbb{B}^q$. A sequence $(z_n) \subset \mathbb{B}^q$ converging to $\zeta \in \partial\mathbb{B}^q$ is said to be *restricted* at ζ if $\langle z_n, \zeta \rangle \to 1$ non-tangentially in $\mathbb{D}$, while it is said to be *special* at ζ if

$$\lim_{n \to \infty} k_{\mathbb{B}^q}(z_n, \langle z_n, \zeta \rangle \zeta) = 0.$$

We say that f has *restricted K-limit* L at ζ (we write $\angle_K \lim_{z \to \zeta} f(z) = L$) if for every special and restricted sequence (z_n) converging to ζ we have that $f(z_n) \to L$.

One can show that

$$K\text{-}\lim_{z \to \zeta} f(z) = L \Longrightarrow \angle_K \lim_{z \to \zeta} f(z) = L,$$

but the converse implication is not true in general.

Definition 4.2. A point $\zeta \in \partial\mathbb{B}^q$ such that $K\text{-}\lim_{z \to \zeta} f(z) = \zeta$ and

$$\liminf_{z \to \zeta} \frac{1 - \|f(z)\|}{1 - \|z\|} = \lambda < +\infty$$

is called a *boundary regular fixed point*, and λ is called its *dilation*.

The following result [20] generalizes the Denjoy–Wolff theorem in the unit disc.

Theorem 4.3. *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be holomorphic. Assume that f admits no fixed points in $\mathbb{B}^q$. Then there exists a point $p \in \partial\mathbb{B}^q$, called the* Denjoy–Wolff point *of f, such that (f^n) converges uniformly on compact subsets to the constant map $z \mapsto p$. The Denjoy–Wolff point of f is a boundary regular fixed point and its dilation λ is smaller than or equal to* 1.

Remark 4.4. Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a holomorphic self-map without fixed points, and let λ be the dilation at its Denjoy–Wolff fixed point. Then by [4, Proposition 5.8] the divergence rate of f satisfyies

$$c(f) = -\log \lambda.$$

Definition 4.5. A holomorphic self-map $f\colon \mathbb{B}^q \to \mathbb{B}^q$ is called

(1) *elliptic* if it admits a fixed point $z \in \mathbb{B}^q$,
(2) *parabolic* if it admits no fixed points $z \in \mathbb{B}^q$, and its dilation at the Denjoy–Wolff point is equal to 1,
(3) *hyperbolic* if it admits no fixed points $z \in \mathbb{B}^q$, and its dilation at the Denjoy–Wolff point is strictly smaller than 1.

If $s_1(z) > 0$ for all $z \in \mathbb{B}^q$, then we say that f is *nonzero-step*.

The next result generalizes Theorem 0.1 to the unit ball.

Theorem 4.6. *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a hyperbolic holomorphic self-map, with dilation λ at its Denjoy–Wolff point $p \in \partial\mathbb{B}^q$. Then there exist*

(1) *an integer k such that $1 \leq k \leq q$,*
(2) *a hyperbolic automorphism $\varphi\colon \mathbb{H}^k \to \mathbb{H}^k$ of the form*

$$\varphi(z,w) = \left(\frac{1}{\lambda}z, \frac{e^{it_1}}{\sqrt{\lambda}}w_1, \ldots, \frac{e^{it_{k-1}}}{\sqrt{\lambda}}w_{k-1}\right), \tag{4.1}$$

where $t_j \in \mathbb{R}$ for $1 \leq j \leq k-1$,
(3) *a holomorphic mapping $h\colon \mathbb{B}^q \to \mathbb{H}^k$ with $K\text{-}\lim_{x\to p} h(x) = \infty$,*

such that the triple $(\mathbb{H}^k, h, \varphi)$ is a canonical Kobayashi hyperbolic model for f.

Proof. Since $\mathbb{B}^q$ is cocompact and Kobayashi hyperbolic, by Theorem 3.6 there exists a canonical Kobayashi hyperbolic semi-model (Z, ℓ, τ) for f. Since Z is a holomorphic retract of $\mathbb{B}^q$, it is biholomorphic to $\mathbb{B}^k$ for some $0 \leq k \leq q$ (see e.g. [1, Corollary 2.2.16]). By Remark 4.4 and by (2) of Theorem 3.6, we have $c(\tau) = c(f) = -\log\lambda$, hence $k \geq 1$ and τ is a hyperbolic automorphism with dilation λ at its Denjoy–Wolff point. Thus there exists (see e.g. [1, Proposition 2.2.11]) a biholomorphism $\gamma\colon Z \to \mathbb{H}^k$ such that $\varphi := \gamma\circ\tau\circ\gamma^{-1}$ is of the form (4.1). Setting $h := \gamma \circ \ell$ we have that $(\mathbb{H}^k, h, \varphi)$ is also a canonical Kobayashi hyperbolic semi-model for f. By [4, Proposition 5.11], we have $K\text{-}\lim_{x\to p} h(x) = \infty$.

□

Corollary 4.7. *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a hyperbolic holomorphic self-map, with dilation λ at its Denjoy–Wolff point $p \in \partial\mathbb{B}^q$. Then there exists a holomorphic mapping $\vartheta\colon \mathbb{B}^q \to \mathbb{H}$ solving the Valiron equation (0.2).*

Proof. Let $(\mathbb{H}^k, h, \varphi)$ be the canonical Kobayashi hyperbolic semi-model given by Theorem 4.6. Let $\pi_1\colon \mathbb{H}^k \to \mathbb{H}$ be the projection $\pi_1(z,w) = z$. Then $\left(\mathbb{H}, \vartheta := \pi_1 \circ h, x \mapsto \frac{1}{\lambda}x\right)$ is a semi-model for f, and thus ϑ solves the Valiron equation (0.2).

□

Remark 4.8. If $q = 1$, then the following uniqueness result holds [11]: any holomorphic solution of the Valiron equation (0.2) is a positive multiple of a given solution $\vartheta\colon \mathbb{H} \to \mathbb{H}$.

If $q \geq 2$, the situation is quite different. It is easy to see that the solutions of (0.2) are all the holomorphic mappings of the form $\Gamma \circ h$, where $(\mathbb{H}^k, h, \varphi)$ is the canonical Kobayashi hyperbolic semi-model given by Theorem 4.6, and $\Gamma\colon \mathbb{H}^k \to \mathbb{H}$ is a holomorphic function such that

$$\Gamma \circ \varphi = \frac{1}{\lambda}\Gamma. \tag{4.2}$$

Notice that for all $z \in \mathbb{H}$,

$$\Gamma\left(\frac{1}{\lambda}z, 0\right) = \frac{1}{\lambda}\Gamma(z, 0),$$

which by a result of Heins [19] implies that $\Gamma(z, 0) = az$ for some $a > 0$ (and thus $\Gamma(\mathbb{H}^k) = \mathbb{H}$). Thus if $k = 1$ we obtain again a uniqueness result: any holomorphic solution of (0.2) is a positive multiple of a given solution $\vartheta\colon \mathbb{H}^q \to \mathbb{H}$.

Assume now that $k \geq 2$. The function Γ is unique up to positive multiples on the slice $\{w = 0\}$ of $\mathbb{H}^k$, but is far from being unique on $\mathbb{H}^k \setminus \{w = 0\}$. This can be seen, for example, in the following way. If $\gamma\colon \mathbb{H}^k \to \mathbb{H}^k$ is a holomorphic self-map which commutes with the hyperbolic automorphism φ, then clearly $\Gamma \coloneqq \pi_1 \circ \gamma$ satisfies (4.2). The family of holomorphic mappings of the form $\pi_1 \circ \gamma$ is large, as shown (and made precise) in [16, Theorem 2.5].

Recall the following result on the Abel equation in the unit disc.

Theorem 4.9 (Pommerenke [28])**.** *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a parabolic nonzero-step holomorphic self-map. Then there exists a model $(\mathbb{H}, h, z \mapsto z \pm 1)$ for f.*

The essential uniqueness of the intertwining mapping in the previous theorem is proved in [27]. The next result gives a generalization of this result to the unit ball.

Theorem 4.10. *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a parabolic nonzero-step holomorphic self-map with Denjoy–Wolff point $p \in \partial\mathbb{B}^q$. Then there exist*

1. *an integer k such that $1 \leq k \leq q$,*
2. *a parabolic automorphism $\varphi\colon \mathbb{H}^k \to \mathbb{H}^k$ of the form*
$$\varphi(z, w) = (z \pm 1, e^{it_1}w_1, \dots e^{it_{k-1}}w_{k-1}), \tag{4.3}$$
where $t_j \in \mathbb{R}$ for $1 \leq j \leq k-1$, or of the form
$$\varphi(z, w) = (z - 2w_1 + i, w_1 - i, e^{it_2}w_2, \dots e^{it_{k-1}}w_{k-1}), \tag{4.4}$$
where where $t_j \in \mathbb{R}$ for $2 \leq j \leq k-1$,
3. *a holomorphic mapping $h\colon \mathbb{B}^q \to \mathbb{H}^k$ with $\angle_K\text{-}\lim_{z\to 0} h(z) = \infty$,*

such that the triple $(\mathbb{H}^k, h, \varphi)$ is a canonical Kobayashi hyperbolic model for f.

Proof. Since $\mathbb{B}^q$ is cocompact and Kobayashi hyperbolic, by Theorem 3.6 there exists a canonical Kobayashi hyperbolic semi-model (Z, ℓ, τ) for f. Since Z is a holomorphic retract of $\mathbb{B}^q$, it is biholomorphic to $\mathbb{B}^k$ for some $0 \leq k \leq q$. Let $z \in Z$, $x \in \mathbb{B}^q$, and $n \geq 0$ such that $\tau^{-n}(\ell(x)) = z$. Then, by (1) of Theorem 3.6,

$$k_Z(z, \tau(z)) = s_1(z) > 0.$$

Hence $k \geq 1$, and τ is not elliptic. By Remark 4.4 and by (2) of Theorem 3.6, we have $c(\tau) = c(f) = 0$. Hence τ is parabolic. There exists (see e.g. [17]) a biholomorphism $\gamma\colon Z \to \mathbb{H}^k$ such that $\varphi := \gamma \circ \tau \circ \gamma^{-1}$ is of the form (4.3) or of the form (4.4). Setting $h := \gamma \circ \ell$ we have that $(\mathbb{H}^k, h, \varphi)$ is also a canonical Kobayashi hyperbolic semi-model for f. By [4, Proposition 5.11], we have $\angle_K\text{-}\lim_{x\to p} h(x) = \infty$. □

Part 2. Backward iteration

5. Preliminaries

Definition 5.1. Let X be a complex manifold. We call *backward (non-autonomous) holomorphic dynamical system* on X any family $(f_{n,m}\colon X \to X)_{m\geq n\geq 0}$ of holomorphic self-maps such that for all $m \geq u \geq n \geq 0$, we have

$$f_{n,u} \circ f_{u,m} = f_{n,m}.$$

For all $n \geq 0$ we denote $f_{n,n+1}$ also by f_n. A backward holomorphic dynamical system $(f_{n,m}\colon X \to X)_{m\geq n\geq 0}$ is called *autonomous* if $f_n = f_0$ for all $n \geq 0$. Clearly in this case $f_{n,m} = f_0^{m-n}$.

Remark 5.2. Any family of holomorphic self-maps $(f_n\colon X \to X)_{n\geq 0}$ determines a backward holomorphic dynamical system $(f_{n,m}\colon X \to X)$ in the following way: for all $n \geq 0$, set $f_{n,n} = \mathsf{id}$, and for all $m > n \geq 0$, set

$$f_{n,m} = f_n \circ \cdots \circ f_{m-1}.$$

Definition 5.3. Let X be a complex manifold, and let $(f_{n,m}\colon X \to X)$ be a backward holomorphic dynamical system. An *inverse limit* for $(f_{n,m})$ is a pair (Θ, V_n) where Θ is a set and $(V_n\colon \Theta \to X)_{n\geq 0}$ is a family of mappings such that

$$f_{n,m} \circ V_m = V_n, \quad \forall\, m \geq n \geq 0,$$

satisfying the following universal property: if Q is a set and if $(g_n\colon Q \to X)$ is a family of mappings satisfying

$$f_{n,m} \circ g_m = g_n, \quad \forall\, m \geq n \geq 0,$$

then there exists a unique mapping $\Gamma\colon Q \to \Theta$ such that

$$g_n = V_n \circ \Gamma, \quad \forall\, n \geq 0.$$

Remark 5.4. The inverse limit is essentially unique, in the following sense. Let (Θ, V_n) and (Q, g_n) be two inverse limits for $(f_{n,m})$. Then there exists a bijective mapping $\Gamma\colon Q \to \Theta$ such that

$$g_n = V_n \circ \Gamma, \quad \forall\, n \geq 0.$$

Definition 5.5. Let X be a complex manifold, and let $(f_{n,m}\colon X \to X)$ be a backward holomorphic dynamical system. A *backward orbit* for $(f_{n,m})$ is a sequence $(x_n)_{n\geq 0}$ in X such that, for all $m \geq n \geq 0$,

$$f_{n,m}(x_m) = x_n.$$

Remark 5.6. An inverse limit for $(f_{n,m})$ is easily constructed. We define Θ as the set of all backward orbits for $(f_{n,m})$. We define a family of mappings $(V_n\colon \Theta \to X)_{n\geq 0}$ in the following way. Let $\beta = (x_m)_{m\geq 0}$ be a backward orbit. Then for all $n \geq 0$,

$$V_n(\beta) = x_n.$$

It is easy to see that (Θ, V_n) is an inverse limit for $(f_{n,m})$.

Definition 5.7. Let X be a complex manifold and let $(f_{n,m}\colon X \to X)_{m\geq n\geq 0}$ be a backward holomorphic dynamical system. Let (Θ, V_n) be the inverse limit of the inverse system $(X, f_{n,m})$. We define an equivalence relation $\sim$ on Θ in the following way. The backward orbits (z_n) and (w_n) are equivalent if and only if the non-decreasing sequence $(k_X(z_n, w_n))_{n\geq 0}$ is bounded. The class of the backward orbit (z_n) will be denoted by $[z_n]$.

Lemma 5.8. *Let X be a complex manifold, and let $(f_{n,m}\colon X \to X)$ be a backward holomorphic dynamical system. Let Z be a complex manifold and let $(\alpha_n\colon Z \to X)$ be a sequence of holomorphic mappings such that $f_{n,m} \circ \alpha_m = \alpha_n$ for all $m \geq n \geq 0$. Then $(\alpha_n(z)) \sim (\alpha_n(w))$ for all $z, w \in Z$.*

Proof. It follows since $k_X(\alpha_n(z), \alpha_n(w)) \leq k_Z(z, w)$ for all $n \geq 0$. □

We now introduce a modified version of the inverse limit for $(f_{n,m})$ which is more suited for our needs.

Definition 5.9. Let X be a complex manifold. Let $(f_{n,m}\colon X \to X)$ be a backward holomorphic dynamical system. We call *canonical inverse limit associated with the class* $[y_n] \in \Theta/_\sim$ for $(f_{n,m})$ a pair (Z, α_n) where Z is a complex manifold and $(\alpha_n\colon Z \to X)$ is a sequence of holomorphic mappings such that

(1) $f_{n,m} \circ \alpha_m = \alpha_n$, for all $m \geq n \geq 0$,
(2) $(\alpha_n(z)) \in [y_n]$ for some (and hence for any) $z \in Z$,

which satisfies the following universal property: if Q is a complex manifold and if $(g_n\colon Q \to X)$ is a family of holomorphic mappings satisfying

(1') $f_{n,m} \circ g_m = g_n$, for all $m \geq n \geq 0$,
(2') $(g_n(q)) \in [y_n]$ for some (and hence for any) $q \in Q$,

then there exists a unique holomorphic mapping $\Gamma\colon Q \to Z$ such that

$$g_n = \alpha_n \circ \Gamma, \quad \forall\, n \geq 0.$$

Proposition 5.10. *The canonical inverse limit for $(f_{n,m})$ associated with the class $[y_n] \in \Theta/_\sim$ is unique in the following sense. Let (Z, α_n) and (Q, g_n) be two canonical inverse limit for $(f_{n,m})$ associated with the same class $[y_n]$. Then there exists a biholomorphism $\Gamma\colon Q \to Z$ such that*

$$g_n = \alpha_n \circ \Gamma, \quad \forall\, n \geq 0.$$

Proof. There exist holomorphic mappings $\Gamma\colon Q \to Z$ and $\Xi\colon Z \to Q$ such that for all $n \geq 0$, we have $g_n = \alpha_n \circ \Gamma$ and $\alpha_n = g_n \circ \Xi$. Thus the holomorphic mapping $\Gamma \circ \Xi\colon Z \to Z$ satisfies

$$\alpha_n \circ \Gamma \circ \Xi = \alpha_n, \quad \forall\, n \geq 0,$$

By the universal property of the canonical inverse limit associated with the class $[y_n] \in \Theta/_\sim$, this implies that $\Gamma \circ \Xi = \mathsf{id}_Z$. Similarly, we obtain $\Xi \circ \Gamma = \mathsf{id}_Q$. □

6. Non-autonomous iteration

Let X be a complete Kobayashi hyperbolic complex manifold. Let $(f_{n,m}\colon X \to X)_{m\geq n\geq 0}$ be a backward holomorphic dynamical system, and assume that it admits a relatively compact backward orbit $(y_m)_{m\geq 0}$.

Remark 6.1. The class $[y_n] \in \Theta/_\sim$ coincides with the subset of Θ defined by all relatively compact backward orbits of $(f_{n,m})$.

Remark 6.2. Let $K \subset X$ be a compact subset such that $\{y_m\}_{m\geq 0} \subset K$. It follows that, for all fixed $n \geq 0$,

$$f_{n,m}(K) \cap K \neq \varnothing \quad \forall\, m \geq n. \tag{6.1}$$

The sequence $(f_{0,m}\colon X \to X)_{m\geq 0}$ is not compactly divergent by (6.1), and since X is taut, there exists a subsequence $(f_{0,m_{k_0}})_{k_0\geq 0}$ converging to a holomorphic self-map $\alpha_0\colon X \to X$. The sequence $(f_{1,m_{k_0}}\colon X \to X)_{k_0\geq 0}$ is not compactly divergent by (6.1), and since X is taut, there exists a subsequence $(f_{1,m_{k_1}})_{k_1\geq 0}$ converging to a holomorphic self-map $\alpha_1\colon X \to X$. Iterating this procedure we obtain a family of holomorphic self-maps $(\alpha_n\colon X \to X)$ satisfying for all $m \geq n \geq 0$,

$$f_{n,m} \circ \alpha_m = \alpha_n. \tag{6.2}$$

Notice that for all $n \geq 0$ we have

$$\alpha_n(K) \cap K \neq \varnothing. \tag{6.3}$$

Let now $(m_k)_{k\geq 0}$ be a sequence which for all $j \in \mathbb{N}$ is eventually a subsequence of $(m_{k_j})_{k_j\geq 0}$ (such a sequence exists by a diagonal argument). The sequence of holomorphic self-maps $(\alpha_{m_k}\colon X \to X)_{k\geq 0}$ is not compactly divergent by (6.3), and since X is taut, there exists a subsequence $(\alpha_{m_h})_{h\geq 0}$ converging to a holomorphic self-map $\alpha\colon X \to X$.

Lemma 6.3. *The holomorphic self-map $\alpha\colon X \to X$ is a holomorphic retraction, and for all $n \geq 0$,*

$$\alpha_n \circ \alpha = \alpha_n. \tag{6.4}$$

Proof. Fix $n \geq 0$ and $x \in X$. Then for all $h \geq 0$ such that $m_h \geq n$, we have

$$\alpha_n(x) = f_{n,m_h}(\alpha_{m_h}(x)) \overset{h\to\infty}{\longrightarrow} \alpha_n(\alpha(x)).$$

Thus we have, for all $h \geq 0$,

$$\alpha_{m_h}(\alpha(x)) = \alpha_{m_h}(x).$$

When $h \to \infty$, the left-hand side converges to $\alpha(\alpha(x))$, while the right-hand side converges to $\alpha(x)$. □

Definition 6.4. We denote the closed complex submanifold $\alpha(X)$ by Z.

In what follows we denote the restriction $\alpha_n|_Z$ simply by α_n. Let (Θ, V_n) be the inverse limit of the inverse system $(X, f_{n,m})$. By the universal property of the inverse limit, there exists a mapping $\Psi\colon Z \to \Theta$ such that for all $n \geq 0$,

$$\alpha_n = V_n \circ \Psi.$$

The mapping Ψ is defined in the following way: if $z \in Z$, then $\Psi(z)$ is the backward orbit $(\alpha_m(z))_{m\geq 0}$.

Proposition 6.5. *The mapping $\Psi\colon Z \to \Theta$ is injective and its image is $[y_n]$.*

Proof. Let $z, w \in Z$ and assume that $\Psi(z) = \Psi(w)$. It follows that $\alpha_m(z) = \alpha_m(w)$ for all $m \geq 0$, that is $\alpha(z) = \alpha(w)$. Since α is a retraction, we obtain $z = w$. Hence $\Psi\colon Z \to \Theta$ is injective.

We now show that $\Psi(Z) \subset [y_n]$. If $z \in Z$, we have to show that the sequence $(k_X(\alpha_m(z), y_m))$ is bounded. Since $y_m \in K$ for all $m \geq 0$ and $\alpha_{m_h}(z) \to \alpha(z)$, we have that the subsequence $(k_X(\alpha_{m_h}(z), y_{m_h}))$ is bounded. Since the sequence $(k_X(\alpha_m(z), y_m))$ is non-decreasing, it is bounded too.

Finally, we show that for all $(z_m) \in [y_n]$, there exists $z \in Z$ such that $\alpha_m(z) = z_m$ for all $m \geq 0$. Let thus (z_m) be a backward orbit such that the sequence $(k_X(y_m, z_m))$ is bounded. Clearly, the subsequence $(k_X(y_{m_h}, z_{m_h}))$ is also bounded, and thus there exists a subsequence (z_{m_u}) of (z_{m_h}) converging to a point $z \in X$. It follows that for all $n \geq 0$,

$$z_n = f_{n,m_u}(z_{m_u}) \overset{u\to\infty}{\to} \alpha_n(z).$$

We claim that $z \in Z$. Indeed, letting $u \to \infty$ in the identity $\alpha_{m_u}(z) = z_{m_u}$ we obtain $\alpha(z) = z$. □

Proposition 6.6. *The pair (Z, α_n) is a canonical inverse limit for $(f_{n,m})$ associated with $[y_n]$.*

Proof. Let Q be a complex manifold and let $(g_n\colon Q \to X)$ be a family of holomorphic mappings satisfying

(1) $f_{n,m} \circ g_m = g_n$, for all $m \geq n \geq 0$,
(2) $(g_n(q)) \in [y_n]$ for some (and hence for any) $q \in Q$.

By the universal property of the inverse limit, there exists a unique mapping $\Phi\colon Q \to \Theta$ such that

$$g_n = V_n \circ \Phi, \quad \forall\, n \geq 0.$$

The mapping Φ is defined in the following way: if $q \in Q$, then $\Phi(q)$ is the backward orbit $(g_m(q))_{m\geq 0}$. Property (2) implies that $\Phi(Q) \subset [y_n]$. Set

$$\Gamma := \Psi^{-1} \circ \Phi\colon Q \to Z.$$

For all $n \geq 0$,

$$\alpha_n \circ \Gamma = V_n \circ \Psi \circ \Gamma = V_n \circ \Phi = g_n. \tag{6.5}$$

The uniqueness of the mapping Γ follows easily from the uniqueness of the mapping Φ. The mapping Γ acts in the following way: if $q \in Q$, then $\Gamma(q) \in Z$ is uniquely defined by

$$\alpha_m(\Gamma(q)) = g_m(q), \quad \forall\, m \geq 0. \tag{6.6}$$

We now prove that Γ is holomorphic. Recall that the sequence $(\alpha_{m_h}\colon Z \to X)_{h\geq 0}$ converges uniformly on compact subsets to id_Z. By Remark 6.1, the sequence $(g_m\colon Q \to X)$ is not compactly divergent. Since X is taut, the sequence $(g_{m_h}\colon Q \to X)$ admits a subsequence $(g_{m_u}\colon Q \to X)$ converging uniformly on compact subsets to a holomorphic mapping $g\colon Q \to X$. Thus taking the limit in both sides of

$$\alpha_{m_u} \circ \Gamma = g_{m_u},$$

as $m_u \to \infty$, we have $\Gamma = g$, which implies that Γ is holomorphic. □

Proposition 6.7. *We have*
$$\lim_{m\to\infty} \alpha_m^* k_X = k_Z,$$
and
$$\lim_{n\to\infty} \alpha_m^* \kappa_X = \kappa_Z.$$

Proof. Let $z, w \in Z$. We have
$$\lim_{m_h\to\infty} k_X(\alpha_{m_h}(z), \alpha_{m_h}(w)) = k_X(\alpha(z), \alpha(w)) = k_X(z,w) = k_Z(z,w).$$
where the last identity follows from the fact that Z is a holomorphic retract of X. The first statement follows since the sequence $(k_X(\alpha_m(z), \alpha_m(w)))_{m\geq 0}$ is non-decreasing. The proof of the second statement is similar. □

Theorem 6.8. *Let X a cocompact Kobayashi hyperbolic complex manifold, and let $(f_{n,m}\colon X \to X)_{m\geq n\geq 0}$ be a backward dynamical system. Let (y_n) be a backward orbit. Then there exists a canonical inverse limit (Z, α_n) for $(f_{n,m})$ associated with $[y_n]$, where Z is a holomorphic retract of X. Moreover,*
$$\lim_{m\to\infty} \alpha_m^* k_X = k_Z, \quad \textit{and} \quad \lim_{m\to\infty} \alpha_m^* \kappa_X = \kappa_Z. \tag{6.7}$$

Proof. Let $K \subset X$ be a compact subset such that $X = \mathrm{Aut}(X)\cdot K$. For all $n \geq 0$, let $h_n \in \mathrm{Aut}(X)$ be such that $h_n^{-1}(y_n) \in K$. For all $m \geq n \geq 0$ set $\tilde f_{n,m} = h_n^{-1} \circ f_{n,m} \circ h_m$. It is easy to see that $(\tilde f_{n,m}\colon X \to X)$ is a forward holomorphic dynamical system with a relatively compact backward orbit $(\tilde y_n \coloneqq h_n^{-1}(y_n))$. We can now apply Proposition 6.6 to $(\tilde f_{n,m}\colon X \to X)$, obtaining a canonical inverse limit $(Z, \tilde\alpha_n)$ for $(\tilde f_{n,m})$ associated with $[\tilde y_n]$, where Z is a holomorphic retract of X. For all $n \geq 0$ set $\alpha_n \coloneqq h_n \circ \tilde\alpha_n$. Clearly
$$f_{n,m} \circ \alpha_m = \alpha_n, \quad \forall\, m \geq n \geq 0.$$
Let Q be a complex manifold and let $(g_n\colon Q \to X)$ be a family of holomorphic mappings satisfying
$$f_{n,m} \circ g_m = g_n, \quad \forall\, m \geq n \geq 0.$$
For all $n \geq 0$ set $\tilde g_n \coloneqq h_n^{-1} \circ g_n$. Then for all $m \geq n \geq 0$,
$$\tilde f_{n,m} \circ \tilde g_m = \tilde f_{n,m} \circ h_m^{-1} \circ g_m = h_n^{-1} \circ f_{n,m} \circ g_m = \tilde g_n.$$
By the universal property of the canonical inverse limit $(Z, \tilde\alpha_n)$ we obtain a holomorphic mapping $\Gamma\colon Q \to Z$ such that
$$\tilde g_n = \tilde \alpha_n \circ \Gamma, \quad \forall\, n \geq 0.$$
Hence $g_n = \alpha_n \circ \Gamma$ for all $n \geq 0$.

Finally, (6.7) follows from Proposition 6.7, since for all $n \geq 0$ the automorphism $h_n\colon X \to X$ is an isometry for k_X and κ_X.

□

Remark 6.9. Let (Θ, V_n) be the inverse limit of the inverse system $(X, f_{n,m})$. Let (y_n) be a backward orbit and let (Z, α_n) be the canonical inverse limit associated with (y_n) given by Theorem 6.8. By the universal property of the inverse limit, there exists a mapping $\Psi\colon Z \to \Theta$ such that
$$\alpha_n = V_n \circ \Psi, \quad \forall\, n \geq 0.$$

It is easy to see that Ψ is injective and that $\Psi(Z) = [y_n]$. In particular, for all $n \geq 0$,

$$\alpha_n(Z) = V_n([y_n]).$$

7. Autonomous iteration

Definition 7.1. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. A *pre-model* for f is a triple (Λ, h, φ) such that Λ is a complex manifold, $h\colon \Lambda \to X$ is a holomorphic mapping and $\varphi\colon \Lambda \to \Lambda$ is an automorphism such that

$$f \circ h = h \circ \varphi.$$

The mapping h is called the *intertwining mapping*.

Let (Λ, h, φ) and (Z, ℓ, τ) be two pre-models for f. A *morphism of pre-models* $\hat{\eta}\colon (\Lambda, h, \varphi) \to (Z, \ell, \tau)$ is given by a holomorphic mapping $\eta\colon \Lambda \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
\Lambda \arrow[rr, "h"] \arrow[dr, "\eta"] \arrow[dd, "\varphi"'] & & X \arrow[dd, "f"] \\
& Z \arrow[ur, "\ell"] \arrow[dd, "\tau"] & \\
\Lambda \arrow[rr, "h"] \arrow[dr, "\eta"'] & & X \\
& Z. \arrow[ur, "\ell"'] &
\end{tikzcd}$$

If the mapping $\eta\colon \Lambda \to Z$ is a biholomorphism, then we say that $\hat{\eta}\colon (\Lambda, h, \varphi) \to (Z, \ell, \tau)$ is an *isomorphism of pre-models*. Notice that then $\eta^{-1}\colon Z \to \Lambda$ induces a morphism $\hat{\eta}^{-1}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$.

Definition 7.2. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. Let (y_n) be a backward orbit for f. Let (Z, ℓ, τ) be a semi-model for f such that for some (and hence for any) $z \in Z$ we have $(\ell(\tau^{-n}(z))) \in [y_n]$. We say that (Z, ℓ, τ) is a *canonical pre-model associated with* $[y_n]$ for f if for any pre-model (Λ, h, φ) for f such that for some (and hence for any) $x \in \Lambda$ we have $(h(\varphi^{-n}(x))) \in [y_n]$, there exists a unique morphism of pre-models $\hat{\eta}\colon (\Lambda, h, \varphi) \to (Z, \ell, \tau)$.

Remark 7.3. If (Z, ℓ, τ) and (Λ, h, φ) are two canonical pre-models for f associated with the same class $[y_n]$, then they are isomorphic.

Lemma 7.4. *Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. Let (y_n) be a backward orbit. If there exists a canonical pre-model (Z, ℓ, τ) for f associated with $[y_n]$, then every backward orbit $(w_n) \in [y_n]$ has bounded step.*

Proof. Let $z \in Z$. The backward orbit $(\ell(\tau^{-n}(z)))$ has bounded step since for all $n \geq 0$,

$$k_X(\ell(\tau^{-n}(z)), \ell(\tau^{-n-1}(z))) \leq k_Z(\tau^{-n}(z), \tau^{-n-1}(z)) = k_Z(z, \tau(z)).$$

Let $(w_n) \in [y_n]$. Since for all $n \geq 0$,

$$k_X(w_n, w_{n+1}) \leq k_X(w_n, \ell(\tau^{-n}(z))) + k_X(\ell(\tau^{-n}(z)), \ell(\tau^{-n-1}(z))) + k_X(\ell(\tau^{-n-1}(z)), w_{n+1}),$$

it follows that (w_n) has also bounded step. $\square$

Theorem 7.5. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $f\colon X \to X$ be a holomorphic self-map. Let (y_n) be a backward orbit with bounded step. Then there exists a canonical pre-model (Z, ℓ, τ) for f associated with $[y_n]$, where Z is a holomorphic retract of X. Moreover, the following holds:*

1. *$\ell(Z) = V_0([y_n])$,*
2. *if $\alpha_m \coloneqq \ell \circ \tau^{-m}$ for all $m \geq 0$, then*
$$\lim_{m\to\infty} \alpha_m^* k_X = k_Z, \quad \lim_{m\to\infty} \alpha_m^* \kappa_X = \kappa_Z,$$
3. *if β is a backward orbit in the class $[y_n]$,*
$$c(\tau) = \lim_{m\to\infty} \frac{\sigma_m(\beta)}{m} = \inf_{m\in\mathbb{N}} \frac{\sigma_m(\beta)}{m}.$$

Proof. Let $(f_{n,m}\colon X \to X)$ be the autonomous dynamical system defined by $f_{n,m} = f^{m-n}$. By Theorem 6.8, there exist a holomorphic retract Z of X and a family of holomorphic mappings $(\alpha_n\colon Z \to X)$ such that the pair (Z, α_n) is a canonical inverse limit associated with $[y_n]$. The sequence of holomorphic mappings $(\beta_n \coloneqq f \circ \alpha_n\colon Z \to X)$ satisfies, for all $m \geq n \geq 0$,
$$f_{n,m} \circ \beta_m = f^{m-n} \circ f \circ \alpha_m = f \circ \alpha_n = \beta_n.$$
Let $z \in Z$ be the unique point such that $\alpha_m(z) = y_m$ for all $m \geq 0$. Then for all $m \geq 1$,
$$k_X(\beta_m(z), y_m) = k_X(\alpha_{m-1}(z), y_m) = k_X(y_{m-1}, y_m),$$
which is bounded since by assumption the backward orbit (y_n) has bounded step. By the universal property of the canonical inverse limit associated with $[y_n]$ there exists a holomorphic self-map $\tau\colon Z \to Z$ such that for all $n \geq 0$,
$$\alpha_n \circ \tau = f \circ \alpha_n.$$

We claim that τ is a holomorphic automorphism. Set for all $n \geq 0$, $\gamma_n \coloneqq \alpha_{n+1}$. For all $m \geq n \geq 0$,
$$f_{n,m} \circ \gamma_m = f^{m-n} \circ \alpha_{m+1} = \alpha_{n+1} = \gamma_n.$$
Let $z \in Z$ be the unique point such that $\alpha_m(z) = y_m$ for all $m \geq 0$. For all $m \geq 0$,
$$k_X(\gamma_m(z), y_m) = k_X(\alpha_{m+1}(z), y_m) = k_X(y_{m+1}, y_m),$$
which is bounded since by assumption the backward orbit (y_n) has bounded step. Thus there exists a holomorphic self-map $\delta\colon Z \to Z$ such that $\alpha_n \circ \delta = \alpha_{n+1}$ for all $n \geq 0$. For all $n \geq 0$ we have
$$\alpha_n \circ \tau \circ \delta = f \circ \alpha_n \circ \delta = f \circ \alpha_{n+1} = \alpha_n,$$
and
$$\alpha_n \circ \delta \circ \tau = \alpha_{n+1} \circ \tau = \alpha_n.$$
By the universal property of the canonical inverse limit associated with $[y_n]$ we have that τ is a holomorphic automorphism and $\delta = \tau^{-1}$. Since for all $n \geq 0$,
$$\alpha_n \circ \tau^n = f^n \circ \alpha_n = \alpha_0,$$

it follows that

$$\alpha_n = \alpha_0 \circ \tau^{-n}.$$

Set $\ell := \alpha_0$. We claim that the triple (Z, ℓ, τ) is a canonical pre-model for f associated with $[y_n]$. Indeed, let (Λ, h, φ) be a pre-model for f such that for some (and hence for any) $x \in \Lambda$ we have $h(\varphi^{-n}(x)) \in [y_n]$. For all $n \geq 0$, let $\lambda_n := h \circ \varphi^{-n}$. Then by the universal property of the canonical inverse limit associated with $[y_n]$ there exists a holomorphic mapping $\eta\colon \Lambda \to Z$ such that for all $n \geq 0$ we have $\alpha_n \circ \eta = \lambda_n$, that is

$$\ell \circ \tau^{-n} \circ \eta = h \circ \varphi^{-n}.$$

Notice that this implies $\ell \circ \eta = h$, and if $n \geq 0$,

$$\alpha_n \circ \tau^{-1} \circ \eta \circ \varphi = h \circ \varphi^{-n-1} \circ \varphi = \lambda_n.$$

Thus by the universal property of the canonical Kobayashi hyperbolic direct limit, $\eta = \tau^{-1} \circ \eta \circ \varphi$. Hence the mapping $\eta\colon \Lambda \to Z$ gives a morphism of pre-models $\hat{\eta}\colon (\Lambda, h, \varphi) \to (Z, \ell, \tau)$.

Property (1) follows from Remark 6.9. Property (2) follows from (6.7). We now prove Property (3). Let $\beta := (w_n)$ be a backward orbit $[y_n]$, and let $z \in Z$ be the unique point such that $\alpha_n(z) = w_n$ for all $n \geq 0$. Then by Property (2) the backward m-step $\sigma_m(\beta)$ satisfies

$$\sigma_m(\beta) = \lim_{n\to\infty} k_X(\alpha_n(z), \alpha_{n+m}(z)) = \lim_{n\to\infty} k_X(\alpha_n(z), \alpha_n(\tau^{-m}(z))) = k_Z(z, \tau^{-m}(z)).$$

Notice that $k_Z(z, \tau^{-m}(z)) = k_Z(z, \tau^m(z))$. We have

$$c(\tau) = \lim_{m\to\infty} \frac{k_Z(z, \tau^m(z))}{m} = \lim_{m\to\infty} \frac{\sigma_m(\beta)}{m},$$

and

$$c(\tau) = \inf_{m\in\mathbb{N}} \frac{k_Z(z, \tau^m(z))}{m} = \inf_{m\in\mathbb{N}} \frac{\sigma_m(\beta)}{m}.$$

□

8. The unit ball

Definition 8.1. Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a holomorphic self-map. Let $\zeta \in \partial\mathbb{B}^q$ be a boundary regular fixed point. The *stable subset* of f at ζ is defined as the subset consisting of all $z \in \mathbb{B}^q$ such that there exists a backward orbit with bounded step starting at z and converging to ζ. We denote it by $\mathcal{S}(\zeta)$.

Clearly $\mathcal{S}(\zeta)$ coincides with the union of all backward orbits in $\mathbb{B}^q$ with bounded step converging to ζ.

Definition 8.2. Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a holomorphic self-map. A *boundary repelling fixed point* $\zeta \in \partial\mathbb{B}^q$ is a boundary regular fixed point with dilation $\lambda > 1$.

The next result generalizes Theorem 0.2 to the unit ball.

Theorem 8.3. *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a holomorphic self-map and let $\zeta \in \partial\mathbb{B}^q$ be a boundary repelling fixed point with dilation $1 < \lambda < \infty$. Let (y_n) be a backward orbit with bounded step which converges to ζ. Define μ by*

$$\mu := \lim_{m\to\infty} e^{\frac{\sigma_m(\beta)}{m}} \geq \lambda,$$

where $\beta \in [y_n]$. Then μ does not depend on $\beta \in [y_n]$ and there exist

(1) *an integer k such that $1 \leq k \leq q$,*
(2) *a hyperbolic automorphism $\varphi\colon \mathbb{H}^k \to \mathbb{H}^k$ with dilation μ at its unique repelling point ∞, of the form*
$$\varphi(z,w) = \left(\frac{1}{\mu}z, \frac{e^{it_1}}{\sqrt{\mu}}w_1, \dots, \frac{e^{it_{k-1}}}{\sqrt{\mu}}w_{k-1}\right), \tag{8.1}$$
where $t_j \in \mathbb{R}$ for $1 \leq j \leq k-1$,
(3) *a holomorphic mapping $h\colon \mathbb{H}^k \to \mathbb{B}^q$ with K-$\lim_{z\to\infty} h(z) = \zeta$,*

such that $(\mathbb{H}^k, h, \varphi)$ is a canonical pre-model for f associated with $[y_n]$, and
$$h(\mathbb{H}^k) = V_0([y_n]) \subset \mathcal{S}(\zeta).$$
If $[y_n]$ contains backward orbit whose convergence to ζ is special and restricted, then $\mu = \lambda$.

Proof. Since $\mathbb{B}^q$ is cocompact and Kobayashi hyperbolic, by Theorem 7.5 there exists a canonical pre-model (Z, ℓ, τ) for f associated with $[y_n]$. Since Z is a holomorphic retract of $\mathbb{B}^q$, it is biholomorphic to $\mathbb{B}^k$ for some $0 \leq k \leq q$. By (3) of Theorem 7.5, if β is a backward orbit in the class $[y_n]$,
$$\mu = \lim_{m\to\infty} e^{\frac{\sigma_m(\beta)}{m}} = e^{c(\tau)}.$$
In particular, μ does not depend on $\beta \in [y_n]$.

We claim that $\mu \geq \lambda$. Let $n \geq 0$. Since λ^n is the dilation at ζ of the mapping f^n, we have, for any $w \in \mathbb{B}^q$ (see e.g. [1]),
$$n \log \lambda = \liminf_{z\to\zeta} (k_{\mathbb{B}^q}(w,z) - k_{\mathbb{B}^q}(w, f^n(z))).$$
Since
$$k_{\mathbb{B}^q}(w,z) - k_{\mathbb{B}^q}(w, f^n(z)) \leq k_{\mathbb{B}^q}(z, f^n(z)),$$
we have that $n \log \lambda \leq \sigma_n(\beta)$, that is, $\lambda \leq e^{\frac{\sigma_n(\beta)}{n}}$. Thus $\mu \geq \lambda$.

The automorphism τ is hyperbolic since the dilation at its Denjoy–Wolff point is equal to $e^{-c(\tau)}$ and
$$e^{-c(\tau)} = \frac{1}{\mu} \leq \frac{1}{\lambda} < 1.$$

There exists (see e.g. [1, Proposition 2.2.11]) a biholomorphism $\gamma\colon Z \to \mathbb{H}^k$ such that $\varphi \coloneqq \gamma \circ \tau \circ \gamma^{-1}$ is of the form (8.1). Setting $h \coloneqq \ell \circ \gamma^{-1}$ we have that $(\mathbb{H}^k, h, \varphi)$ is also a canonical pre-model for f associated with $[y_n]$.

We now address the regularity at ∞ of the intertwining mapping h. Let (z_n, w_n) be a backward orbit in $\mathbb{H}^k$ for τ. Then (z_n, w_n) converges to ∞ and there exists $C > 0$ such that
$$k_{\mathbb{H}^k}((z_n, w_n), (z_{n+1}, w_{n+1})) \leq C, \quad \text{and} \quad k_{\mathbb{H}^k}((z_n, w_n), (z_n, 0)) \leq C.$$
Clearly $g(z_n, w_n)$ is a backward orbit for f which converges to $\zeta \in \partial\mathbb{B}^q$. Then [4, Theorem 5.6] yields the result.

Theorem 7.5 yields that $h(\mathbb{H}^k) = V_0([y_n])$. Let $x \in V_0([y_n])$. Then there exists a backward orbit $(w_n) \in [y_n]$ starting at x, which clearly converges to ζ. By Lemma 7.4 the backward orbit (w_n) has bounded step, and thus $V_0([y_n]) \subset \mathcal{S}(\zeta)$.

Let $\beta := (w_n)$ be a special and restricted backward orbit in $[y_n]$ converging to ζ. Then the same proof as in [3, Proposition 4.12] shows that

$$\log \mu = \lim_{m\to\infty} \frac{\sigma_m(\beta)}{m} = \log \lambda.$$

$\square$

We leave the following open questions.

Question 8.4. With notations from the previous theorem, does the identity $\lambda = \mu$ always hold?

Question 8.5. Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a holomorphic self-map and let $\zeta \in \partial\mathbb{B}^q$ be a boundary repelling fixed point with dilation $1 < \lambda < \infty$. By [23, Lemma 3.1], if ζ is isolated from other boundary repelling fixed points with dilation less or equal than λ, then $\mathcal{S}(\zeta) \neq \varnothing$. Is the same true if the point ζ is not isolated?

Question 8.6. Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a parabolic self-map and let $p \in \partial\mathbb{B}^q$ be its Denjoy–Wolff point. Let (y_n) be a backward orbit with bounded step which converges to p. Let (Z, ℓ, τ) be a canonical pre-model associated with $[y_n]$. Clearly τ cannot be elliptic. Is τ parabolic? In the unit disc, it follows from [26, Theorem 1.12] that this is true.

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